Exercise 7E

1 $\overrightarrow{XY} = \overrightarrow{XW} + \overrightarrow{WY} = \mathbf{b} - \mathbf{a}$
$\overrightarrow{YZ} = \overrightarrow{YW} + \overrightarrow{WZ} = \mathbf{c} - \mathbf{b}$
Since $\overrightarrow{XY} = \overrightarrow{YZ}$:
$\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{b}$
$\mathbf{b} + \mathbf{b} = \mathbf{a} + \mathbf{c}$
$\mathbf{a} + \mathbf{c} = 2\mathbf{b}$

2 a i $\overrightarrow{OB} = 2\overrightarrow{OR}$
$= 2\mathbf{r}$

ii $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$ (addition of vectors)
$= -\overrightarrow{OP} + \overrightarrow{OQ}$
$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ}$ (addition of vectors)
$\overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AB}$
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ (addition of vectors)
$= -\overrightarrow{OA} + \overrightarrow{OB}$
$= -2\mathbf{p} + 2\mathbf{r}$
$\therefore \overrightarrow{AQ} = \frac{1}{2}(-2\mathbf{p} + 2\mathbf{r})$
$= -\mathbf{p} + \mathbf{r}$
$\therefore \overrightarrow{OQ} = 2\mathbf{p} + (-\mathbf{p} + \mathbf{r})$
$= \mathbf{p} + \mathbf{r}$
$\therefore \overrightarrow{PQ} = -\mathbf{p} + (\mathbf{p} + \mathbf{r})$
$= \mathbf{r}$

b $\overrightarrow{OB} = 2\mathbf{r}$ and $\overrightarrow{PQ} = \mathbf{r}$
$\Rightarrow \overrightarrow{OB}$ and $\overrightarrow{PQ}$ are parallel.
$\Rightarrow \angle AOB = \angle APQ$ and $\angle ABO = \angle AQP$
(corresponding angles, parallel lines)
Angle A is common to both triangles.
$\Rightarrow \Delta PAQ$ and ΔOAB are similar (three equal angles)

3 a M divides OA in the ratio $2:1$.
$\Rightarrow \overrightarrow{OM} = \frac{2}{3}\mathbf{a}$
Using vector addition:
$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$
$\overrightarrow{AN} = \lambda\overrightarrow{AB}$ (N lies on AB, so $\overrightarrow{AN} = \lambda\overrightarrow{AB}$)
$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
$\overrightarrow{ON} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

3 b $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$
$= \overrightarrow{OM} + \mu\overrightarrow{OB}$ ($\overrightarrow{MN}$ is parallel to $\overrightarrow{OB}$)
$= \frac{2}{3}\mathbf{a} + \mu\mathbf{b}$
But $\overrightarrow{ON} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$
So:
$\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = \frac{2}{3}\mathbf{a} + \mu\mathbf{b}$
$\mathbf{a}(1 - \lambda) + \lambda\mathbf{b} = \frac{2}{3}\mathbf{a} + \mu\mathbf{b}$
$\Rightarrow$ (comparing coefficients of **a** and **b**):
$1 - \lambda = \frac{2}{3}$ and $\lambda = \mu$
so $\lambda = \mu = \frac{1}{3}$ and $\overrightarrow{ON} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$
$\overrightarrow{AN} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$
$\Rightarrow \overrightarrow{NB} = \frac{2}{3}(\mathbf{b} - \mathbf{a})$
$\Rightarrow AN : NB = 1:2$

4 a M is the midpoint of OA, so:
$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA}$
$= \frac{1}{2}\mathbf{a}$
Using vector addition:
$\overrightarrow{MQ} = \overrightarrow{MA} + \overrightarrow{AB} + \overrightarrow{BQ}$
$= \overrightarrow{MA} + \overrightarrow{AB} + \frac{1}{4}\overrightarrow{BC}$
$= \frac{1}{2}\mathbf{a} + \mathbf{c} - \frac{1}{4}\mathbf{a}$
$= \frac{1}{4}\mathbf{a} + \mathbf{c}$
and:
$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
$= -\mathbf{a} + \mathbf{c}$
$= \mathbf{c} - \mathbf{a}$
P lies on AC and MQ, so:
$\overrightarrow{OP} = \overrightarrow{OM} + \lambda\overrightarrow{MQ}$
$= \frac{1}{2}\mathbf{a} + \lambda\left(\frac{1}{4}\mathbf{a} + \mathbf{c}\right)$
and $\overrightarrow{OP} = \overrightarrow{OA} + \mu\overrightarrow{AC}$
$= \mathbf{a} + \mu(\mathbf{c} - \mathbf{a})$
Comparing coefficients of **a** and **c**:
$\frac{1}{2} + \frac{1}{4}\lambda = 1 - \mu$ and $\lambda = \mu$
$\Rightarrow \frac{5}{4}\lambda = \frac{1}{2}$
$\lambda = \mu = \frac{2}{5}$
$\overrightarrow{OP} = \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{c}$

4 b $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$
$= -\mathbf{a} + 0.6\mathbf{a} + 0.4\mathbf{c}$
$= 0.4(\mathbf{c} - \mathbf{a})$

$\overrightarrow{PC} = \overrightarrow{PO} + \overrightarrow{OC}$
$= -0.6\mathbf{c} - 0.4\mathbf{c} + \mathbf{c}$
$= 0.6(\mathbf{c} - \mathbf{a})$

Therefore $\overrightarrow{AP} : \overrightarrow{PC} = 2:3$ as required.

5 a $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$$= -\begin{pmatrix}5\\8\end{pmatrix} + \begin{pmatrix}4\\3\end{pmatrix}$$

$$= \begin{pmatrix}-1\\-5\end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{(-1)^2 + (-5)^2}$$

$$= \sqrt{26}$$

b $\overrightarrow{AC} = -\overrightarrow{OA} + \overrightarrow{OC}$

$$= -\begin{pmatrix}5\\8\end{pmatrix} + \begin{pmatrix}7\\6\end{pmatrix}$$

$$= \begin{pmatrix}2\\-2\end{pmatrix}$$

$$|\overrightarrow{AC}| = \sqrt{2^2 + (-2)^2}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2}$$

c $\overrightarrow{BC} = -\overrightarrow{OB} + \overrightarrow{OC}$

$$= -\begin{pmatrix}4\\3\end{pmatrix} + \begin{pmatrix}7\\6\end{pmatrix}$$

$$= \begin{pmatrix}3\\3\end{pmatrix}$$

$$|\overrightarrow{BC}| = \sqrt{3^2 + 3^2}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

5 d Using the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{(2\sqrt{2})^2 + (\sqrt{26})^2 - (3\sqrt{2})^2}{2(2\sqrt{2})(\sqrt{26})}$$

$$\cos A = \frac{8 + 26 - 18}{4\sqrt{52}}$$

$$\cos A = \frac{16}{8\sqrt{13}}$$

$$\cos A = \frac{2}{\sqrt{13}}$$

$A = 56.3...°$

Using the sine rule:

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin B}{2\sqrt{2}} = \frac{\sin 56.3°}{3\sqrt{2}}$$

$$\sin B = \frac{2\sqrt{2}\sin 56.3°}{3\sqrt{2}}$$

$B = 33.68...°$

$C = 180° - 56° - 34° = 90°$

The angles are 56°, 34° and 90°.

6 a $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PR} + \overrightarrow{RS}$

$\overrightarrow{OP} = \mathbf{a}$

$\overrightarrow{PR} = \frac{1}{3}(-\mathbf{a} + \mathbf{b})$

$\overrightarrow{RS} = 2\overrightarrow{OR} = 2\left(\overrightarrow{OP} + \overrightarrow{PR}\right)$

$= 2\left(\mathbf{a} + \frac{1}{3}(-\mathbf{a} + \mathbf{b})\right)$

$= 2\mathbf{a} - \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

$= \frac{4}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

So $\overrightarrow{OS} = \mathbf{a} + \frac{1}{3}(-\mathbf{a} + \mathbf{b}) + \frac{4}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

$= 2\mathbf{a} + \mathbf{b}$

b $\overrightarrow{TP} = \overrightarrow{TO} + \overrightarrow{OP}$

$= \mathbf{b} + \mathbf{a} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$

$= \frac{1}{3}(-\mathbf{a} + \mathbf{b}) + \frac{4}{3}\mathbf{a} + \frac{2}{3}\mathbf{b} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{TP}$ is parallel and equal to $\overrightarrow{PS}$ and point P is common to both lines, so T, P and S lie on a straight line.

Challenge

a Since X lies on PR, $\overrightarrow{PX} = j\overrightarrow{PR}$

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$$
$$= -\mathbf{a} + \mathbf{b}$$
$$\overrightarrow{PX} = j(-\mathbf{a} + \mathbf{b})$$
$$= -j\mathbf{a} + j\mathbf{b}$$

b $\overrightarrow{PX} = \overrightarrow{PO} + \overrightarrow{OX}$

$$\overrightarrow{OX} = k\overrightarrow{ON}$$
$$\overrightarrow{ON} = \mathbf{a} + \tfrac{1}{2}\mathbf{b}$$
$$\overrightarrow{PX} = -\mathbf{a} + k\left(\mathbf{a} + \tfrac{1}{2}\mathbf{b}\right)$$
$$= (k-1)\mathbf{a} + \tfrac{1}{2}k\mathbf{b}$$

c As $\overrightarrow{PX} = -j\mathbf{a} + j\mathbf{b}$

and $\overrightarrow{PX} = (k-1)\mathbf{a} + \tfrac{1}{2}k\mathbf{b}$

then $-j\mathbf{a} + j\mathbf{b} = (k-1)\mathbf{a} + \tfrac{1}{2}k\mathbf{b}$

The coefficients of **a** and **b** must be the same, so $k - 1 = -j$ and $\tfrac{1}{2}k = j$.

d Solving the equation simultaneously and using substitution:

$$k - 1 = -\tfrac{1}{2}k$$
$$k = \tfrac{2}{3}$$
$$j = \tfrac{1}{3}$$

e $\overrightarrow{PX} = -\tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b}$

As $OPQR$ is a parallelogram, $\overrightarrow{YR} = \overrightarrow{PX}$.
Therefore $\overrightarrow{PX} = \overrightarrow{XY} = \overrightarrow{YR}$, so the line PR is divided into three equal parts.
Therefore, the lines ON and OM divide the diagonal PR into three equal parts.